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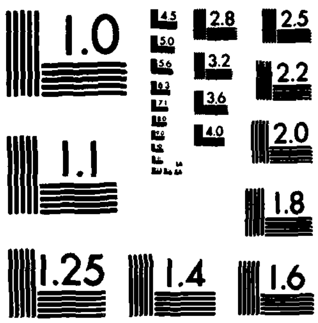

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Properties of Systems which Lead to
Efficient Computation of Reliability

Michael O. Ball* and J. Scott Provan**

Technical Report No. UNC/ORSA/TR-84/16

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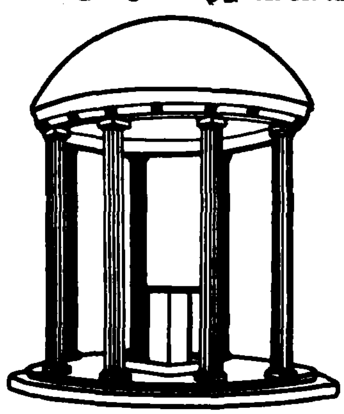

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ABSTRACT

One of the most widely used approaches to computing system reliability is to represent the system structure in terms of a Boolean sum of all minpaths. This expression is then transformed into a sum of disjoint terms. The probability of each term is then summed to obtain the reliability of the system. A key question with respect to the difficulty of this process relates to the ability to transform the initial sum into a sum of disjoint products. In this paper, we show that for the class of shellable systems, there always exists a disjoint product expression with a number of terms equal to the number of minpaths. We provide several examples of shellable systems for which such an expression can be efficiently found.

Introduction

In this paper we consider the property of shellability of a coherent system, and show how it can be used to compute or approximate reliability for these systems. In particular, we show how one can directly evaluate the disjoint products form for reliability by writing this expression as a shelling for the system. We also indicate how the property of shellability can help to construct good approximations to reliability when even the disjoint products method is too cumbersome. We present several applications for which the underlying system is shellable, and give examples to show the shelling.

1. Definitions and Notation

Consider a set N of n components. Each component i is subject to failure with probability p_i, and all components fail independently. A structure function ϕ is given which maps the n-vector $x=(x_1,\ldots,x_n)$ of component indicators ($x_i=1$ if component i operates, and 0 otherwise) into the system status ($\phi(x)=1$ if the system operates and 0 otherwise). We assume that the function ϕ is coherent, that is, $\phi(\underline{0}) = 0$, $\phi(\underline{1}) = 1$ and ϕ is nondecreasing in each component. In this case, the function ϕ can be described completely by giving the set $\{T_1,\ldots,T_r\}$ of minpaths, that is, T_i is a minimal set of components whose operation insures system operation. It follows that $\phi(x)$ equals 1 if and only if the set $\{j:x_j=1\}$ of operating components contains at least one of the minpaths T_i. Thus the event of system operation can be described as the union of events (denoted by plus)

$$R_\phi = (\phi=1) = A_1+\ldots+A_r$$

where A_i represents the event that all components of T_i operate. Unfortunately, one cannot immediately write the probability for this expression because the events are not disjoint. One of the ways to proceed is to write the above expression in disjoint products form, that is, as a union of mutually exclusive events

$$R_\phi = A_1 + \bar{A}_1 A_2 + \ldots + \bar{A}_1\bar{A}_2\ldots\bar{A}_{r-1}A_r$$

where product denotes intersection and bar denotes the complementary event. The probability of system operation can now be written as the sum of probabilities of the events

$$R_{\phi,i} = \bar{A}_1\bar{A}_2\ldots\bar{A}_{i-1}A_i$$

and the problem is reduced to computing the probabilities of each of the $R_{\phi,i}$. Many papers have dealt with effective use of the disjoint product form to compute system reliability (see [1],[2],[3],[6],[11],[12],[13],[16],[17]).

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Unfortunately, computing $P_r(R_{\phi,i})$ can also be quite complex. What we would like to investigate in this paper are systems for which we can write the disjoint product in such a way that the terms $R_{\phi,i}$ can be computed directly as intersections of <u>component events</u>

$$R_{\phi,i} = X_{j_1}\cdots X_{j_\ell}\bar{X}_{j_{\ell+1}}\cdots\bar{X}_{j_k}$$

where X_j is the event that component j operates and $\bar{X}_j$ is the event that component j fails. This leads to the following definition.

<u>Definition</u>: A coherent system is <u>shellable</u> if there is an ordering $T_1, T_2, \ldots, T_r$ of minpaths such that the event $R_{\phi,i}$ can be written as a disjoint product of component events. The corresponding expression for R_ϕ is called a <u>shelling</u> for the system.

It is clear that if a shelling can be obtained for a system, then the probability of system operation can be written immediately as the sum of the probabilities of the events $R_{\phi,i}$, which can be written in turn as a product of component probabilities:

$$P_r(R_{\phi,i}) = (1-p_{j_1})\cdots(1-p_{j_\ell})p_{j_{\ell+1}}\cdots p_{j_k}$$

by using the independence of component events.

<u>Example</u>: Consider the coherent system defined on component set $N = \{1,2,3,4,5\}$ which has the set of minpaths $\{123, 14, 24, 34\}$. (For simplicity, we will use the same notation for T_i and A_i.) This is not shellable in the order given, since the event $(\overline{123})(14) = X_1\bar{X}_2X_4 + X_1X_2\bar{X}_3X_4$ cannot be written as a single disjoint product of component events. The ordering $(14, 24, 34, 123)$, however, <u>is</u> a shelling, for the system operation can now be written in disjoint product form

$$\begin{aligned} R_\phi &= (14) + (\overline{14})(24) + (\overline{14})(\overline{24})(34) \\ &\quad + (\overline{14})(\overline{24})(\overline{34})(123) \\ &= X_1X_4 + \bar{X}_1X_2X_4 + \bar{X}_1\bar{X}_2X_3X_4 \\ &\quad + X_1X_2X_3\bar{X}_4 \end{aligned}$$

The probability of system operation can be written directly as

$$\begin{aligned} P_r(R_\phi) &= (1-p_1)(1-p_4) + p_1(1-p_2)(1-p_4) \\ &\quad + p_1p_2(1-p_3)(1-p_4) \\ &\quad + (1-p_1)(1-p_2)(1-p_3)p_4. \end{aligned}$$

There is a second way that shellability can aid in reliability computation, and this does not depend on knowing a specific shelling at all. It turns out that shellability provides combinatorial structure to a system which restricts the number of operating configurations (x such that $\phi(x) = 1$) with a given number of operating components. This makes it possible to obtain <u>approximations</u> to system reliability, under the assumption of equal component failure probabilities, which are considerably more accurate than those that can be obtained without having the property of shellability. Moreover, there are very efficient methods of obtaining these approximations, and so such approximation techniques are useful when the system is too complex to allow exact computation of reliability. The details of this procedure are somewhat involved, and the reader is urged to consult [5] and [14] for details.

Shellability, therefore, is a convenient property for a system to have, in terms of facilitating reliability computation. It is <u>not</u> a property held by all systems. Take, <u>for</u> example the four component coherent system whose minpaths are $\{12, 34\}$. Neither of the events $(\overline{12})(34) = \bar{X}_1X_3X_4 + X_1\bar{X}_2X_3X_4$ nor $(\overline{34})(12) = X_1X_2\bar{X}_3 + X_1X_2X_3\bar{X}_4$ can be written as a single product of component events, and so this system is not shellable. There are, however, a large number of classic reliability problems, with important applications, for which the underlying system is shellable. For the remainder of this paper, we give examples of these systems, along with the appropriate shellings.

2. <u>Matroid Systems</u>

Matroids make up a class of coherent binary systems that have been studied extensively in a variety of contexts. In particular, they have been shown to be especially powerful (see [8], [9]) incharacterizing combinatorial optimization problems for which efficient solution algorithms exist. Matroids are shellable ([15], Corollary 3.2.2) and thus provide a class of systems for which the properties described in Section 1 hold. In this section, we describe two important classes of systems that are matroids and that are of interest in the study of system reliability.

A. Graph Connectedness: Much of the study of system reliability has employed network models. One measure of system reliability used in the study of computer communications networks is

$$P_r[\text{operating paths exist between all node pairs}].$$

In the case where only arcs fail, the underlying coherent binary system is what is known as a graphic matroid. For the graph illustrated in Figure 1, the

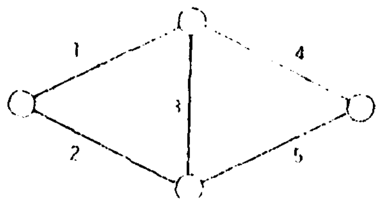

Figure 1

minpaths are $\{124, 125, 134, 135, 145, 234, 235, 245\}$, representing the spanning trees in the graph. These produce the shelling

$$
\begin{aligned}
R_\phi = {} & 124 + (\overline{124})(125) + (\overline{124})(\overline{125})(134) \\
& + (\overline{124})(\overline{125})(\overline{134})(135) \\
& + (\overline{124})(\overline{125})(\overline{134})(\overline{135})(145) \\
& + (\overline{124})(\overline{125})(\overline{134})(\overline{135})(\overline{145})(234) \\
& + (\overline{124})(\overline{125})(\overline{134})(\overline{135})(\overline{145})(\overline{234})(235) \\
& + (\overline{124})(\overline{125})(\overline{134})(\overline{135})(\overline{145})(\overline{234})(\overline{235})(245) \\
= {} & x_1x_2x_4 + x_1x_2\bar{x}_4x_5 + x_1\bar{x}_2x_3x_4 \\
& + x_1\bar{x}_2x_3\bar{x}_4x_5 + x_1\bar{x}_2\bar{x}_3x_4x_5 + \bar{x}_1x_2x_3x_4 \\
& + \bar{x}_1x_2x_3\bar{x}_4x_5 + \bar{x}_1x_2\bar{x}_3x_4x_5
\end{aligned}
$$

This class of systems has been studied extensively, not only because of its importance from an application standpoint, but also because of the rich structure it contains.

B. Personnel Assignment Systems: We now discuss a second class of systems, which we believe has important practical applications. This system relates to the reliability of a personnel system. Figure 2 illustrates the problem. The left hand set of nodes corresponds to jobs critical to the survivial of an operation. The right hand set of nodes corresponds to personnel. An arc is inserted between a right hand and left

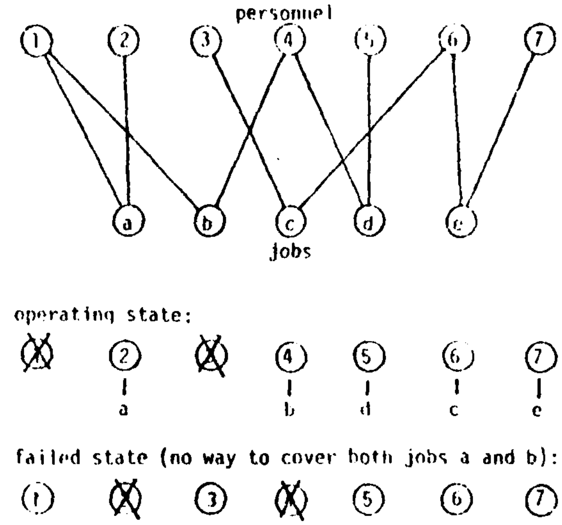

Figure 2: A personnel system

hand node if the job can be filled by the corresponding person. This model is, of course, the classic assignment model. In a reliability setting the components correspond to personnel. The failure of a person corresponds to that person being unable to work, and the system operates any time the set of operating persons can cover all the required jobs. Thus, the measure of system reliability is

$$Pr[\text{the operating persons can cover all required jobs}].$$

Note that given the set of failed persons, the determination of whether the system can operate is not completely straightforward. In fact, it involves the solution of an assignment problem.

This system is what is known as a transversal matroid. If there are n jobs, then minpaths consist of sets of n persons with the property that there exists a way of assigning those n persons to the n jobs. For the system of Figure 2, the minpaths are $\{12346, 12347, 12356, 12357, 12467, 12567, 13456, 13457, 14567, 23456, 23457, 24567\}$, and these produce shelling (in the order given)

$$
\begin{aligned}
R_\phi = {} & x_1x_2x_3x_4x_6 + x_1x_2x_3x_4\bar{x}_6x_7 \\
& + x_1x_2x_3\bar{x}_4x_5x_6 + x_1x_2x_3\bar{x}_4x_5\bar{x}_6x_7 \\
& + x_1x_2\bar{x}_3x_4x_6x_7 + x_1x_2\bar{x}_3\bar{x}_4x_5x_6x_7 \\
& + x_1\bar{x}_2x_3x_4x_5x_6 + x_1\bar{x}_2x_3x_4x_5\bar{x}_6x_7 \\
& + x_1\bar{x}_2\bar{x}_3x_4x_5x_6x_7 + \bar{x}_1x_2x_3x_4x_5x_6 \\
& + \bar{x}_1x_2x_3x_4x_5\bar{x}_6x_7 + \bar{x}_1x_2\bar{x}_3x_4x_5x_6x_7.
\end{aligned}
$$

3. Linear Systems

A second broad category of shellable systems includes those systems which can be represented as linear systems. Specifically, suppose the components of a system correspond to real variables which are subject to some set of linear constraints (equalities or inequalities). Component failures correspond to setting the corresponding variables to zero, and the system then operates if the remaining variables can still take values which satisfy the constraints. We need to make a further assumption that the linear system is nondegenerate (in the linear programming sense), although the two examples given here will be nondegenerate. It can be shown ([7], Lemma 2.1) that these systems are shellable, and so the results of Section 1 apply.

A. Reachability: Suppose we are given a directed network G whose arcs are subject to failure. The system operates if the operating arcs admit directed paths from a given source node s to all other nodes of G. An example is given by the network in Figure 3. The linear representation

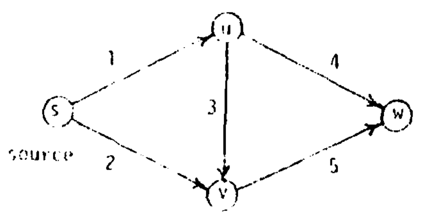

Figure 3

of this system is

$$\begin{aligned} y_1 \quad - y_3 - y_4 \quad &= 1 \\ y_2 + y_3 \quad - y_5 &= 1 \\ y_4 + y_5 &= 1 \end{aligned}$$

$$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ y_4 \geq 0,\ y_5 \geq 0$$

Since any solution which satisfies these constraints must have the property that the arcs corresponding to nonzero components admit paths from s to u, v, and w. One can check that the minpaths for this system are {124, 125, 134, 135} and these produce the shelling

$$\begin{aligned} R_\phi &= (124) + (\overline{124})(125) \\ &\quad + (\overline{124})(\overline{125})(134) \\ &\quad + (\overline{124})(\overline{125})(\overline{134})(135) \\ &= x_1 x_2 x_4 + x_1 x_2 \bar{x}_4 x_5 + x_1 \bar{x}_2 x_3 x_4 \\ &\quad + x_1 \bar{x}_2 x_3 \bar{x}_4 x_5 \end{aligned}$$

B. Transportation Problems: Here we are given a set of m_1 supply points with supply a_i available at supply point i, and m_2 demand points, with demand b_j required at demand point j. We assume that total supply equals total demand, but that no proper subset of supply and demand points can have supply equal to demand. Each link (i,j) between supply point i and demand point j is subject to failure, and the system operates if the operating links are sufficient to ship from available supplies to meet the demands. As an example we take the three supply, two demand system with $a_1=2$, $a_2=6$, $a_3=7$, $b_1=5$, and $b_2=10$. The linear representation of this system is

Component	1	2	3	4	5	6
Link	(1,1)	(1,2)	(2,1)	(2,2)	(3,1)	(3,2)

$$\begin{aligned} y_1 + y_2 \quad\quad\quad\quad &= 2 \\ y_3 + y_4 \quad\quad &= 6 \\ y_5 + y_6 &= 7 \\ y_1 \quad + y_3 \quad + y_5 \quad &= 5 \\ y_2 \quad + y_4 \quad + y_6 &= 10 \end{aligned}$$

$$y_1 \geq 0\ \ y_2 \geq 0\ \ y_3 \geq 0\ \ y_4 \geq 0\ \ y_5 \geq 0\ \ y_6 \geq 0$$

and the minpaths are {1346, 1456, 2346, 2356}. These produce the shelling

$$\begin{aligned} R_\phi &= (1346) + (\overline{1346})(1456) \\ &\quad + (\overline{1346})(\overline{1456})(2346) \\ &\quad + (\overline{1346})(\overline{1456})(\overline{2346})(2456) \\ &= x_1 x_3 x_4 x_5 + x_1 \bar{x}_3 x_4 x_5 x_6 + \bar{x}_1 x_2 x_3 x_4 x_6 \\ &\quad + \bar{x}_1 x_2 \bar{x}_3 x_4 x_5 x_6. \end{aligned}$$

Conclusion

We end the paper with a word on algorithms which find shellings for these systems, since good shelling algorithms will lead naturally to good algorithms for computing system reliability. Shelling algorithms do exist for the problems given here that run in time polynomial in the number of minpaths. Those for matroid complexes can be construed from [4], and those for linear systems from [7], Lemma 2.1. Neither of these papers, however, give direct shelling algorithms, and considerably more efficient algorithms can be obtained by considering the extra structure contained in the systems presented. These algorithms will be covered in a subsequent paper.

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